PART 6

OPTIONS

CHAPTER 21

# Valuing Options

**In the last** chapter we introduced you to call and put options. Call options give the owner the right to buy an asset at a specified exercise price; put options give the right to sell. We also took the first step toward understanding how options are valued. The value of a call option depends on five variables:

1. The higher the price of the asset, the more valuable an option to buy it.
2. The lower the price that you must pay to exercise the call, the more valuable the option.
3. You do not need to pay the exercise price until the option expires. This delay is most valuable when the interest rate is high.
4. If the stock price is below the exercise price at maturity, the call is valueless regardless of whether the price is $1 below or $100 below. However, for every dollar that the stock price rises above the exercise price, the option holder gains an additional dollar. Thus, the value of the call option increases with the volatility of the stock price.
5. Finally, a long-term option is more valuable than a short-term option. A distant maturity delays the point at which the holder needs to pay the exercise price and increases the chance of a large jump in the stock price before the option matures.

In this chapter we show how these variables can be combined into an exact option-valuation model—a formula we can plug numbers into to get a definite answer. We first describe a simple way to value options, known as the binomial model. We then introduce the Black–Scholes formula for valuing options. Finally, we provide a checklist showing how these two methods can be used to solve a number of practical option problems.

The most efficient way to value most options is to use a computer. But in this chapter we will work through some simple examples by hand. We do so because unless you understand the basic principles behind option valuation, you are likely to make mistakes in setting up an option problem and you won't know how to interpret the computer's answer and explain it to others.

In the last chapter we looked at the put and call options on Google stock. In this chapter we stick with that example and show you how to value the Google options. But remember *why* you need to understand option valuation. It is not to make a quick buck trading on an options exchange. It is because many capital budgeting and financing decisions have options embedded in them. We discuss a variety of these options in subsequent chapters.

## 21-1 A Simple Option-Valuation Model

### Why Discounted Cash Flow Won't Work for Options

For many years economists searched for a practical formula to value options until Fischer Black and Myron Scholes finally hit upon the solution. Later we will show you what they found, but first we should explain why the search was so difficult.

Our standard procedure for valuing an asset is to (1) figure out expected cash flows and (2) discount them at the opportunity cost of capital. Unfortunately, this is not practical for options. The first step is messy but feasible, but finding *the* opportunity cost of capital is impossible, because the risk of an option changes every time the stock price moves.

When you buy a call, you are taking a position in the stock but putting up less of your own money than if you had bought the stock directly. Thus, an option is always riskier than the underlying stock. It has a higher beta and a higher standard deviation of return.

How much riskier the option is depends on the stock price relative to the exercise price. A call option that is in the money (stock price greater than exercise price) is safer than one that is out of the money (stock price less than exercise price). Thus a stock price increase raises the option's price *and* reduces its risk. When the stock price falls, the option's price falls *and* its risk increases. That is why the expected rate of return investors demand from an option changes day by day, or hour by hour, every time the stock price moves.

We repeat the general rule: The higher the stock price is relative to the exercise price, the safer is the call option, although the option is always riskier than the stock. The option's risk changes every time the stock price changes.

## Constructing Option Equivalents from Common Stocks and Borrowing

If you've digested what we've said so far, you can appreciate why options are hard to value by standard discounted-cash-flow formulas and why a rigorous option-valuation technique eluded economists for many years. The breakthrough came when Black and Scholes exclaimed, "Eureka! We have found it![1] The trick is to set up an *option equivalent* by combining common stock investment and borrowing. The net cost of buying the option equivalent must equal the value of the option."

We'll show you how this works with a simple numerical example. We'll travel back to September 2008 and consider a six-month call option on Google stock with an exercise price of $430. We'll pick a day when Google stock was also trading at $430, so that this option is *at the money*. The short-term, risk-free interest rate was 3% per year, or about 1.5% for six months.

To keep the example as simple as possible, we assume that Google stock can do only two things over the option's six-month life: either the price will fall by a quarter to $322.5 or rise by one-third to $573.33.

If Google's stock price falls to $322.50, the call option will be worthless, but if the price rises to $573.33, the option will be worth 573.33 − 430 = $143.33. The possible payoffs to the option are therefore:

| | Stock Price = $322.50 | Stock Price = $573.33 |
|---|---|---|
| 1 call option | $0 | $143.33 |

Now compare these payoffs with what you would get if you bought 4/7 Google shares and borrowed $181.56 from the bank:[2]

| | Stock Price = $322.50 | Stock Price = $573.33 |
|---|---|---|
| 4/7 shares | $184.29 | $327.62 |
| Repayment of loan + interest | −184.29 | −184.29 |
| Total payoff | $ 0 | $143.33 |

[1] We do not know whether Black and Scholes, like Archimedes, were sitting in bathtubs at the time.

[2] The amount that you need to borrow from the bank is simply the present value of the difference between the payoffs from the option and the payoffs from 4/7 shares. In our example, amount borrowed = ((4/7) × 322.50 − 0)/1.015 = ((4/7) × 573.33 − 143.33)/1.015 = $181.56.

Notice that the payoffs from the levered investment in the stock are identical to the payoffs from the call option. Therefore, the law of one price tells us that both investments must have the same value:

$$\text{Value of call} = \text{value of } (4/7) \text{ shares} - \$181.56 \text{ bank loan}$$
$$= 430 \times (4/7) - 181.56 = 64.15$$

Presto! You've valued a call option.

To value the Google option, we borrowed money and bought stock in such a way that we exactly replicated the payoff from a call option. This is called a **replicating portfolio.** The number of shares needed to replicate one call is called the **hedge ratio** or **option delta.** In our Google example one call is replicated by a levered position in 4/7 shares. The option delta is, therefore, 4/7, or about .571.

How did we know that Google's call option was equivalent to a levered position in 4/7 shares? We used a simple formula that says:

$$\text{Option delta} = \frac{\text{spread of possible option prices}}{\text{spread of possible share prices}} = \frac{143.33 - 0}{573.33 - 322.50} = \frac{143.33}{250.83} = \frac{4}{7}$$

You have learned not only to value a simple option but also that you can replicate an investment in the option by a levered investment in the underlying asset. Thus, if you can't buy or sell a call option on an asset, you can create a homemade option by a replicating strategy—that is, you buy or sell delta shares and borrow or lend the balance.

**Risk-Neutral Valuation** Notice why the Google call option should sell for \$64.15. If the option price is higher than \$64.15, you could make a certain profit by buying 4/7 shares of stock, selling a call option, and borrowing \$181.56. Similarly, if the option price is less than \$64.15, you could make an equally certain profit by selling 4/7 shares, buying a call, and lending the balance. In either case there would be an arbitrage opportunity.[3]

If there's a possible arbitrage profit, everyone scurries to take advantage of it. So when we said that the option price had to be \$64.15 or there would be an arbitrage opportunity, we did not have to know anything about investor attitudes to risk. The option price cannot depend on whether investors detest risk or do not care a jot.

This suggests an alternative way to value the option. We can *pretend* that all investors are *indifferent* about risk, work out the expected future value of the option in such a world, and discount it back at the risk-free interest rate to give the current value. Let us check that this method gives the same answer.

If investors are indifferent to risk, the expected return on the stock must be equal to the risk-free rate of interest:

$$\text{Expected return on Google stock} = 1.5\% \text{ per six months}$$

We know that Google stock can either rise by 33.3% to \$573.33 or fall by 25% to \$322.50. We can, therefore, calculate the probability of a price rise in our hypothetical risk-neutral world:

$$\text{Expected return} = [\text{probability of rise} \times 33.3]$$
$$+ [(1 - \text{probability of rise}) \times (-25)]$$
$$= 1.5\%$$

Therefore,

$$\text{Probability of rise} = .4543, \text{ or } 45.43\%$$

[3] Of course, you don't get seriously rich by dealing in 4/7 shares. But if you multiply each of our transactions by a million, it begins to look like real money.

Notice that this is *not* the *true* probability that Google stock will rise. Since investors dislike risk, they will almost surely require a higher expected return than the risk-free interest rate from Google stock. Therefore the true probability is greater than .4543.

The general formula for calculating the risk-neutral probability of a rise in value is:

$$p = \frac{\text{interest rate} - \text{downside change}}{\text{upside change} - \text{downside change}}$$

In the case of Google stock:

$$p = \frac{.015 - (-.25)}{.333 - (-.25)} = .4543$$

We know that if the stock price rises, the call option will be worth $143.33; if it falls, the call will be worth nothing. Therefore, if investors are risk-neutral, the expected value of the call option is:

$$[\text{Probability of rise} \times 143.33] + [(1 - \text{probability of rise}) \times 0]$$
$$= (.4543 \times 143.33) + (.5457 \times 0)$$
$$= \$65.11$$

And the *current* value of the call is:

$$\frac{\text{Expected future value}}{1 + \text{interest rate}} = \frac{65.11}{1.015} = \$64.15$$

Exactly the same answer that we got earlier!

We now have two ways to calculate the value of an option:

1. Find the combination of stock and loan that replicates an investment in the option. Since the two strategies give identical payoffs in the future, they must sell for the same price today.
2. Pretend that investors do not care about risk, so that the expected return on the stock is equal to the interest rate. Calculate the expected future value of the option in this hypothetical *risk-neutral* world and discount it at the risk-free interest rate. This idea may seem familiar to you. In Chapter 9 we showed how you can value an investment either by discounting the expected cash flows at a risk-adjusted discount rate or by adjusting the expected cash flows for risk and then discounting these *certainty-equivalent* flows at the risk-free interest rate. We have just used this second method to value the Google option. The certainty-equivalent cash flows on the stock and option are the cash flows that would be expected in a risk-neutral world.

## Valuing the Google Put Option

Valuing the Google call option may well have seemed like pulling a rabbit out of a hat. To give you a second chance to watch how it is done, we will use the same method to value another option—this time, the six-month Google put option with a $430 exercise price.[4] We continue to assume that the stock price will either rise to $573.33 or fall to $322.50.

If Google's stock price rises to $573.33, the option to sell for $430 will be worthless. If the price falls to $322.50, the put option will be worth $430 − 322.50 = $107.50. Thus the payoffs to the put are:

| | Stock Price = $322.50 | Stock Price = $573.33 |
|---|---|---|
| 1 put option | $107.50 | $0 |

[4] When valuing *American* put options, you need to recognize the possibility that it will pay to exercise early. We discuss this complication later in the chapter, but it is unimportant for valuing the Google put and we ignore it here.

We start by calculating the option delta using the formula that we presented above:[5]

$$\text{Option delta} = \frac{\text{spread of possible option prices}}{\text{spread of possible stock prices}} = \frac{0 - 107.50}{573.33 - 322.50}$$
$$= -\frac{3}{7} \text{ or about } -.429$$

Notice that the delta of a put option is always negative; that is, you need to *sell* delta shares of stock to replicate the put. In the case of the Google put you can replicate the option payoffs by *selling* 3/7 Google shares and *lending* $242.09. Since you have sold the share short, you will need to lay out money at the end of six months to buy it back, but you will have money coming in from the loan. Your net payoffs are exactly the same as the payoffs you would get if you bought the put option:

| | Stock Price = $322.50 | Stock Price = $573.33 |
|---|---|---|
| Sale of 3/7 shares | − $138.22 | − $245.72 |
| Repayment of loan + interest | +245.72 | +245.72 |
| Total payoff | $107.50 | $ 0 |

Since the two investments have the same payoffs, they must have the same value:

$$\text{Value of put} = -(3/7) \text{ shares} + \$242.09 \text{ bank loan}$$
$$= -(3/7) \times 430 + \$242.09 = \$57.82$$

**Valuing the Put Option by the Risk-Neutral Method** Valuing the Google put option with the risk-neutral method is a cinch. We already know that the probability of a rise in the stock price is .4543. Therefore the expected value of the put option in a risk-neutral world is:

$$[\text{Probability of rise} \times 0] + [(1 - \text{probability of rise}) \times 107.50]$$
$$= (.4543 \times 0) + (.5457 \times 107.50)$$
$$= \$58.66$$

And therefore the *current* value of the put is:

$$\frac{\text{Expected future value}}{1 + \text{interest rate}} = \frac{58.66}{1.015} = \$57.80$$

**The Relationship between Call and Put Prices** We pointed out earlier that for European options there is a simple relationship between the value of the call and that of the put.[6]

$$\text{Value of put} = \text{value of call} + \text{present value of exercise price} - \text{share price}$$

Since we had already calculated the value of the Google call, we could also have used this relationship to find the value of the put:

$$\text{Value of put} = 64.15 + \frac{430}{1.015} - 430 = \$57.80$$

Everything checks.

[5] The delta of a put option is always equal to the delta of a call option with the same exercise price minus one. In our example, delta of put = (4/7) − 1 = − 3/7.

[6] *Reminder:* This formula applies only when the two options have the same exercise price and exercise date.

## 21-2 The Binomial Method for Valuing Options

The essential trick in pricing any option is to set up a package of investments in the stock and the loan that will exactly replicate the payoffs from the option. If we can price the stock and the loan, then we can also price the option. Equivalently, we can pretend that investors are risk-neutral, calculate the expected payoff on the option in this fictitious risk-neutral world, and discount by the rate of interest to find the option's present value.

These *concepts* are completely general, but there are several ways to find the replicating package of investments. The example in the last section used a simplified version of what is known as the **binomial method.** The method starts by reducing the possible changes in next period's stock price to two, an "up" move and a "down" move. This assumption that there are just two possible prices for Google stock at the end of six months is clearly fanciful.

We could make the Google problem a trifle more realistic by assuming that there are two possible price changes in each three-month period. This would give a wider variety of six-month prices. And there is no reason to stop at three-month periods. We could go on to take shorter and shorter intervals, with each interval showing two possible changes in Google's stock price and giving an even wider selection of six-month prices.

This is illustrated in Figure 21.1. The top diagram shows our starting assumption: just two possible prices at the end of six months. Moving down, you can see what happens when there are two possible price changes every three months. This gives three possible stock prices when the option matures. In Figure 21.1(*c*) we have gone on to divide the six-month period into 26 weekly periods, in each of which the price can make one of two small moves. The distribution of prices at the end of six months is now looking much more realistic.

We could continue in this way to chop the period into shorter and shorter intervals, until eventually we would reach a situation in which the stock price is changing continuously and there is a continuum of possible future stock prices.

### Example: The Two-Stage Binomial Method

Dividing the period into shorter intervals doesn't alter the basic method for valuing a call option. We can still replicate the call by a levered investment in the stock, but we need to adjust the degree of leverage at each stage. We demonstrate first with our simple two-stage case in Figure 21.1(*b*). Then we work up to the situation where the stock price is changing continuously.

Figure 21.2 is taken from Figure 21.1(*b*) and shows the possible prices of Google stock, assuming that in each three-month period the price will either rise by 22.56% or fall by 18.41%.[7] We show in parentheses the possible values at maturity of a six-month call option with an exercise price of $430. For example, if Google's stock price turns out to be $286.27 in month 6, the call option will be worthless; at the other extreme, if the stock value is $645.90, the call will be worth $645.90 − $430 = $215.90. We haven't worked out yet what the option will be worth *before* maturity, so we just put question marks there for now.

**Option Value in Month 3** To find the value of Google's option today, we start by working out its possible values in month 3 and then work back to the present. Suppose that at the end of three months the stock price is $527.00. In this case investors know that, when the option finally matures in month 6, the stock price will be either $430 or

[7] We explain shortly why we picked these figures.

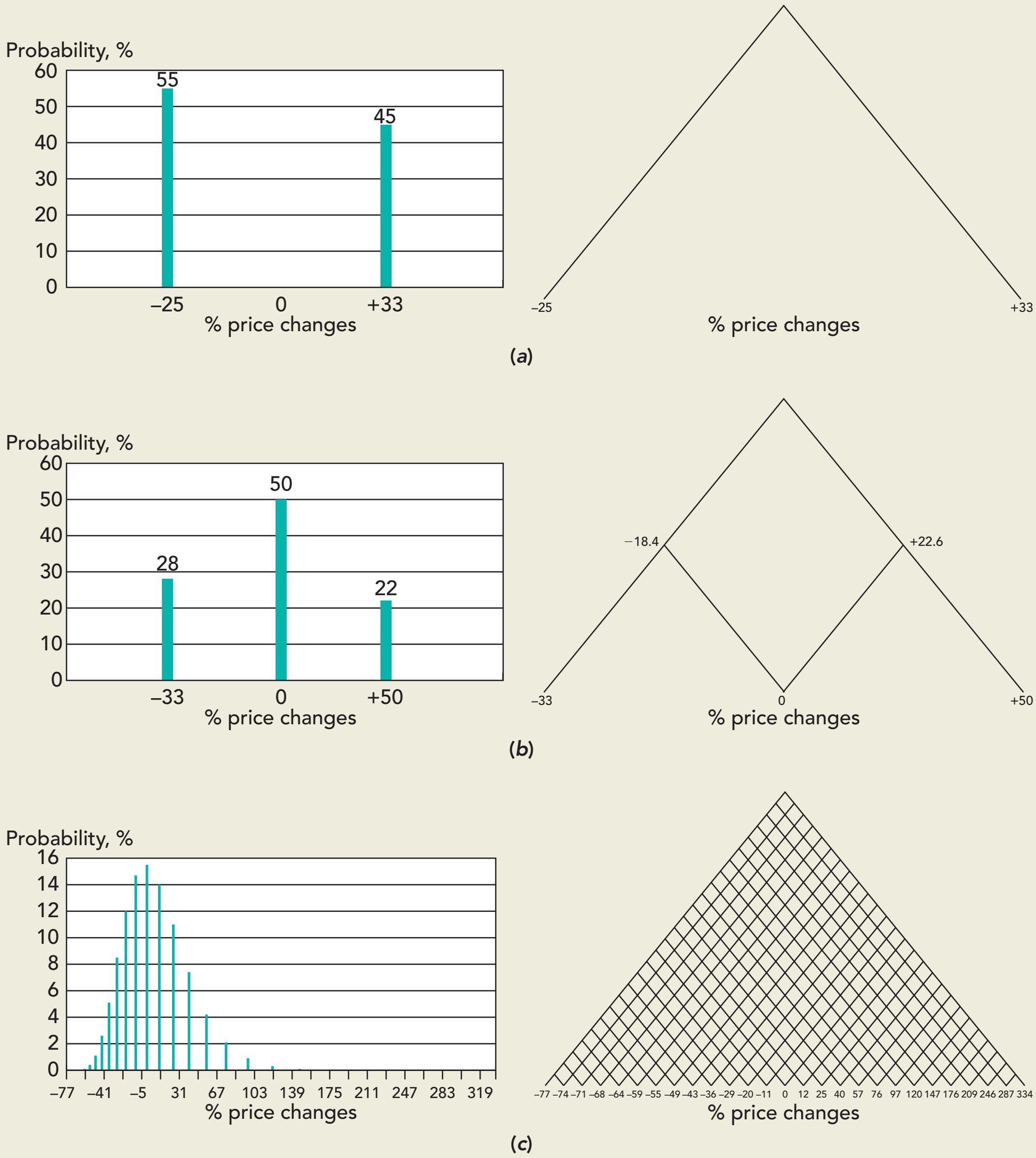



**FIGURE 21.1**

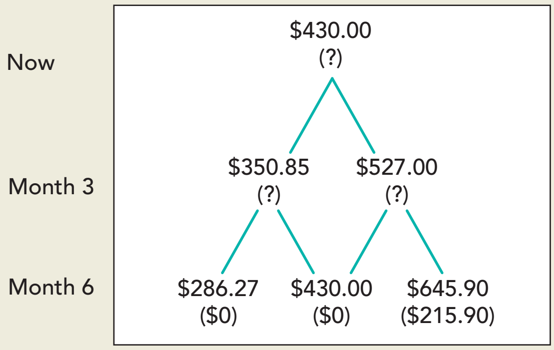
This figure shows the possible six-month price changes for Google stock assuming that the stock makes a single up or down move each six months (Fig. 21.1[*a*]), 2 moves, one every three months (Fig. 21.1[*b*]), or 26 moves, one every week (Fig. 21.1[*c*]). Beside each tree we show a histogram of the possible six-month price changes, assuming investors are risk-neutral.

\$645.89, and the corresponding option price will be \$0 or \$215.89. We can therefore use our simple formula to find how many shares we need to buy in month 3 to replicate the option:

$$\text{Option delta} = \frac{\text{spread of possible option prices}}{\text{spread of possible stock prices}} = \frac{215.89 - 0}{645.89 - 430} = 1.0$$

**FIGURE 21.2**

Present and possible future prices of Google stock assuming that in each three-month period the price will either rise by 22.6% or fall by 18.4%. Figures in parentheses show the corresponding values of a six-month call option with an exercise price of $430.




Now we can construct a leveraged position in delta shares that would give identical payoffs to the option:

| | Month 6 Stock Price = $430 | Month 6 Stock Price = $645.90 |
|---|---|---|
| Buy 1.0 shares | $430 | $645.90 |
| Borrow PV(430) | −430 | −430 |
| Total payoff | $ 0 | $215.90 |

Since this portfolio provides identical payoffs to the option, we know that the value of the option in month 3 must be equal to the price of one share less the $430 loan discounted for three months at 3% per year, about .75% for three months:

$$\text{Value of call in month 3} = \$527.00 - \$430/1.0075 = \$100.20$$

Therefore, if the share price rises in the first three months, the option will be worth $100.20. But what if the share price falls to $350.85? In that case the most that you can hope for is that the share price will recover to $430. Therefore the option is bound to be worthless when it matures and must be worthless at month 3.

**Option Value Today** We can now get rid of two of the question marks in Figure 21.2. Figure 21.3 shows that if the stock price in month 3 is $527.00, the option value is $100.20 and, if the stock price is $350.85, the option value is zero. It only remains to work back to the option value today.

We again begin by calculating the option delta:

$$\text{Option delta} = \frac{\text{spread of possible option prices}}{\text{spread of possible stock prices}} = \frac{100.20 - 0}{527.00 - 350.85} = .5688$$

We can now find the leveraged position in delta shares that would give identical payoffs to the option:

| | Month 3 Stock Price = $350.85 | Month 3 Stock Price = $527.00 |
|---|---|---|
| Buy .5688 shares | $199.56 | $299.76 |
| Borrow PV(199.56) | −199.56 | −199.56 |
| Total payoff | $ 0 | $100.20 |

The value of the Google option today is equal to the value of this leveraged position:

$$\begin{aligned}\text{PV option} &= .5688 \times \text{Share price} - \text{PV}(\$199.56)\\ &= .5688 \times \$430 - \$199.56/1.0075 = \$46.51\end{aligned}$$

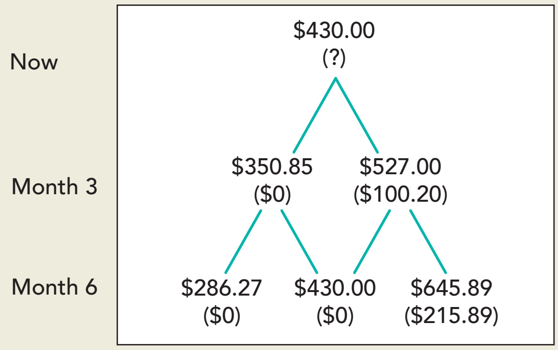



**FIGURE 21.3**

Present and possible future prices of Google stock. Figures in parentheses show the corresponding values of a six-month call option with an exercise price of $430.

## The General Binomial Method

Moving to two steps when valuing the Google call probably added extra realism. But there is no reason to stop there. We could go on, as in Figure 21.1, to chop the period into smaller and smaller intervals. We could still use the binomial method to work back from the final date to the present. Of course, it would be tedious to do the calculations by hand, but simple to do so with a computer.

Since a stock can usually take on an almost limitless number of future values, the binomial method gives a more realistic and accurate measure of the option's value if we work with a large number of subperiods. But that raises an important question. How do we pick sensible figures for the up and down changes in value? For example, why did we pick figures of +22.6% and −18.4% when we revalued Google's option with two subperiods? Fortunately, there is a neat little formula that relates the up and down changes to the standard deviation of stock returns:

$$1 + \text{upside change} = u = e^{\sigma\sqrt{h}}$$
$$1 + \text{downside change} = d = 1/u$$

where,

$e$ = base for natural logarithms = 2.718
$\sigma$ = standard deviation of (continuously compounded) stock returns
$h$ = interval as fraction of a year

When we said that Google's stock price could either rise by 33.3% or fall by 25% over six months ($h = .5$), our figures were consistent with a figure of 40.68% for the standard deviation of annual returns:[8]

$$1 + \text{upside change(6-month interval)} = u = e^{.4068\sqrt{.5}} = 1.333$$
$$1 + \text{downside change} = d = 1/u = 1/1.333 = .75$$

To work out the equivalent upside and downside changes when we divide the period into two three-month intervals ($h = .25$), we use the same formula:

$$1 + \text{upside change(3-month interval)} = u = e^{.4068\sqrt{.25}} = 1.226$$
$$1 + \text{downside change} = d = 1/u = 1/1.226 = .816$$

[8] To find the standard deviation given $u$, we turn the formula around:

$$\sigma = \log(u)/\sqrt{h}$$

where log = natural logarithm. In our example:

$$\sigma = \log(1.333)/\sqrt{.5} = .2877/\sqrt{.5} = .4068$$

**TABLE 21.1** As the number of steps is increased, you must adjust the range of possible changes in the value of the asset to keep the same standard deviation. But you will get increasingly close to the Black–Scholes value of the Google call option.

(*Note:* The standard deviation is $\sigma = .4068$)

| | Change per Interval (%) | | |
|---|---|---|---|
| Number of Steps | Upside | Downside | Estimated Option Value |
| 1 | +33.3 | −25.0 | $64.15 |
| 2 | +22.6 | −18.4 | 46.49 |
| 6 | +12.5 | −11.1 | 50.05 |
| 26 | +5.8 | −5.5 | 51.57 |
| | | Black–Scholes value = | 52.04 |

The center columns in Table 21.1 show the equivalent up and down moves in the value of the firm if we chop the period into six monthly or 26 weekly periods, and the final column shows the effect on the estimated option value. (We explain the Black–Scholes value shortly.)

### The Binomial Method and Decision Trees

Calculating option values by the binomial method is basically a process of solving decision trees. You start at some future date and work back through the tree to the present. Eventually the possible cash flows generated by future events and actions are folded back to a present value.

Is the binomial method *merely* another application of decision trees, a tool of analysis that you learned about in Chapter 10? The answer is no, for at least two reasons. First, option pricing theory is absolutely essential for discounting within decision trees. Discounting expected cash flows doesn't work within decision trees for the same reason that it doesn't work for puts and calls. As we pointed out in Section 21-1, there is no single, constant discount rate for options because the risk of the option changes as time and the price of the underlying asset change. There is no single discount rate inside a decision tree, because if the tree contains meaningful future decisions, it also contains options. The market value of the future cash flows described by the decision tree has to be calculated by option pricing methods.

Second, option theory gives a simple, powerful framework for describing complex decision trees. For example, suppose that you have the option to abandon an investment. The complete decision tree would overflow the largest classroom chalkboard. But now that you know about options, the opportunity to abandon can be summarized as "an American put." Of course, not all real problems have such easy option analogies, but we can often approximate complex decision trees by some simple package of assets and options. A custom decision tree may get closer to reality, but the time and expense may not be worth it. Most men buy their suits off the rack even though a custom-made Armani suit would fit better and look nicer.

## 21-3 The Black–Scholes Formula

Look back at Figure 21.1, which showed what happens to the distribution of possible Google stock price changes as we divide the option's life into a larger and larger number of increasingly small subperiods. You can see that the distribution of price changes becomes increasingly smooth.


**FIGURE 21.4**

As the option's life is divided into more and more subperiods, the distribution of possible stock price changes approaches a lognormal distribution.

If we continued to chop up the option's life in this way, we would eventually reach the situation shown in Figure 21.4, where there is a continuum of possible stock price changes at maturity. Figure 21.4 is an example of a lognormal distribution. The lognormal distribution is often used to summarize the probability of different stock price changes.[9] It has a number of good commonsense features. For example, it recognizes the fact that the stock price can never fall by more than 100%, but that there is some, perhaps small, chance that it could rise by much more than 100%.

Subdividing the option life into indefinitely small slices does not affect the principle of option valuation. We could still replicate the call option by a levered investment in the stock, but we would need to adjust the degree of leverage continuously as time went by. Calculating option value when there is an infinite number of subperiods may sound a hopeless task. Fortunately, Black and Scholes derived a formula that does the trick.[10] It is an unpleasant-looking formula, but on closer acquaintance you will find it exceptionally elegant and useful. The formula is:

$$\text{Value of call option} = \underset{\displaystyle [N(d_1)}{\underset{\uparrow}{[\text{delta}}} \times \underset{\displaystyle P]}{\underset{\uparrow}{\text{share price}]}} - \underset{\displaystyle [N(d_2) \times \text{PV(EX)}]}{\underset{\uparrow}{[\text{bank loan}]}}$$

where,

$$d_1 = \frac{\log[P/\text{PV(EX)}]}{\sigma\sqrt{t}} + \frac{\sigma\sqrt{t}}{2}$$

$$d_2 = d_1 - \sigma\sqrt{t}$$

$N(d)$ = cumulative normal probability density function[11]

EX = exercise price of option; PV(EX) is calculated by discounting at the risk-free interest rate $r_f$

[9] When we first looked at the distribution of stock price changes in Chapter 8, we depicted these changes as normally distributed. We pointed out at the time that this is an acceptable approximation for very short intervals, but the distribution of changes over longer intervals is better approximated by the lognormal.

[10] The pioneering articles on options are F. Black and M. Scholes, "The Pricing of Options and Corporate Liabilities," *Journal of Political Economy* 81 (May–June 1973), pp. 637–654; and R. C. Merton, "Theory of Rational Option Pricing," *Bell Journal of Economics and Management Science* 4 (Spring 1973), pp. 141–183.

[11] That is, $N(d)$ is the probability that a normally distributed random variable $\tilde{x}$ will be less than or equal to $d$. $N(d_1)$ in the Black–Scholes formula is the option delta. Thus the formula tells us that the value of a call is equal to an investment of $N(d_1)$ in the common stock less borrowing of $N(d_2) \times$ PV(EX).

$t$ = number of periods to exercise date
$P$ = price of stock now
$\sigma$ = standard deviation per period of (continuously compounded) rate of return on stock

Notice that the value of the call in the Black–Scholes formula has the same properties that we identified earlier. It increases with the level of the stock price $P$ and decreases with the present value of the exercise price PV(EX), which in turn depends on the interest rate and time to maturity. It also increases with the time to maturity and the stock's variability ($\sigma\sqrt{t}$).

To derive their formula Black and Scholes assumed that there is a continuum of stock prices, and therefore to replicate an option investors must continuously adjust their holding in the stock.[12] Of course this is not literally possible, but even so the formula performs remarkably well in the real world, where stocks trade only intermittently and prices jump from one level to another. The Black–Scholes model has also proved very flexible; it can be adapted to value options on a variety of assets such as foreign currencies, bonds, and commodities. It is not surprising, therefore, that it has been extremely influential and has become the standard model for valuing options. Every day dealers on the options exchanges use this formula to make huge trades. These dealers are not for the most part trained in the formula's mathematical derivation; they just use a computer or a specially programmed calculator to find the value of the option.

## Using the Black–Scholes Formula

The Black–Scholes formula may look difficult, but it is very straightforward to apply. Let us practice using it to value the Google call.

Here are the data that you need:

- Price of stock now = $P$ = 430
- Exercise price = EX = 430
- Standard deviation of continuously compounded annual returns = $\sigma$ = .4068
- Years to maturity = $t$ = .5
- Interest rate per annum = $r_f$ = 3% (or about 1.5% for six months).[13]

Remember that the Black–Scholes formula for the value of a call is

$$[N(d_1) \times P] - [N(d_2) \times \text{PV(EX)}]$$

where,

$$d_1 = \log[P/\text{PV(EX)}]/\sigma\sqrt{t} + \sigma\sqrt{t}/2$$
$$d_2 = d_1 - \sigma\sqrt{t}$$
$N(d)$ = cumulative normal probability function

There are three steps to using the formula to value the Google call:

[12] The important assumptions of the Black-Scholes formula are that (a) the price of the underlying asset follows a lognormal random walk, (b) investors can adjust their hedge continuously and costlessly, (c) the risk-free rate is known, and (d) the underlying asset does not pay dividends.

[13] More precisely, an annually compounded interest rate of 3% is equivalent to a six-month rate of $1.03^{.5} - 1 = .014889$, or 1.4889%. Thus PV(EX) = 430/1.014889 = $423.69.

When valuing options, it is more common to use continuously compounded rates (see Section 2-4). If the annually compounded rate is 3%, the equivalent continuously compounded rate is 2.956%. (The natural log of 1.03 is .02956 and $e^{.02956} = 1.03$.) Using continuous compounding, PV(EX) = $430 \times e^{-.5 \times .02956}$ = $423.69.

If both methods give the same answer, why do we bother to mention the subject here? It is simply because most computer programs for valuing options call for a continuously compounded rate. If you enter an annually compounded rate by mistake, the error will usually be small, but you can waste a lot of time trying to trace it.

**Step 1** Calculate $d_1$ and $d_2$. This is just a matter of plugging numbers into the formula (noting that "log" means *natural* log):

$$\begin{aligned} d_1 &= \log[P/\text{PV(EX)}]/\sigma\sqrt{t} + \sigma\sqrt{t}/2 \\ &= \log[430/(430/1.015)]/(.4068 \times \sqrt{.5}) + .4068 \times \sqrt{.5}/2 \\ &= .1956 \\ d_2 &= d_1 - \sigma\sqrt{t} = .1956 - .4068 \times \sqrt{.5} = -.0921 \end{aligned}$$

**Step 2** Find $N(d_1)$ and $N(d_2)$. $N(d_1)$ is the probability that a normally distributed variable will be less than $d_1$ standard deviations above the mean. If $d_1$ is large, $N(d_1)$ is close to 1.0 (i.e., you can be almost certain that the variable will be less than $d_1$ standard deviations above the mean). If $d_1$ is zero, $N(d_1)$ is .5 (i.e., there is a 50% chance that a normally distributed variable will be below the average).

The simplest way to find $N(d_1)$ is to use the Excel function NORMSDIST. For example, if you enter NORMSDIST(.1956) into an Excel spreadsheet, you will see that there is a .5775 probability that a normally distributed variable will be less than .1956 standard deviations above the mean. Alternatively, you can use a set of normal probability tables such as the present value tables (Appendix Table 6) located on the book Web site at **www.mhhe.com/bma**.

Again you can use the Excel function to find $N(d_2)$. If you enter NORMSDIST(−.0921) into an Excel spreadsheet, you should get the answer .4633. In other words, there is a probability of .4633 that a normally distributed variable will be less than .0921 standard deviations *below* the mean. Alternatively, if you use Appendix Table 6 (again located at **www.mhhe.com/bma** under the Present Value Tables heading), you need to look up the value for +.0921 and subtract it from 1.0:

$$\begin{aligned} N(d_2) &= N(-.0921) = 1 - N(+.0921) \\ &= 1 - .5367 = .4633 \end{aligned}$$

**Step 3** Plug these numbers into the Black–Scholes formula. You can now calculate the value of the Google call:

$$\begin{aligned} &[\text{Delta} \times \text{price}] - [\text{bank loan}] \\ &= [N(d_1) \times P] - [N(d_2) \times \text{PV(EX)}] \\ &= [.5775 \times 430] - [.4633 \times 430/1.015] = 248.32 - 196.29 = 52.04 \end{aligned}$$

In other words, you can replicate the Google call option by investing $248.32 in the company's stock and borrowing $196.29. Subsequently, as time passes and the stock price changes, you may need to borrow a little more to invest in the stock or you may need to sell some of your stock to reduce your borrowing.

## The Risk of an Option

How risky is the Google call option? We have seen that you can exactly replicate a call by a combination of risk-free borrowing and an investment in the stock. So the risk of the option must be the same as the risk of this replicating portfolio. We know that the beta of any portfolio is simply a weighted average of the betas of the separate holdings. So the risk of the option is just a weighted average of the betas of the investments in the loan and the stock.

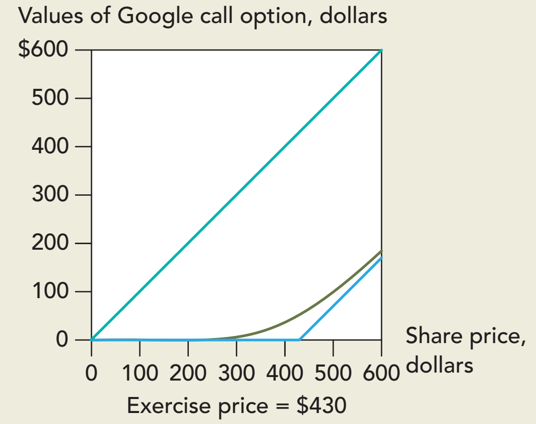
On past evidence the beta of Google stock is $\beta_{stock} = 1.27$; the beta of a risk-free loan is $\beta_{loan} = 0$. You are investing $248.32 in the stock and −$196.29 in the loan. (Notice that the investment in the loan is negative—you are *borrowing* money.) Therefore the beta of the option is $\beta_{option} = (-196.29 \times 0 + 248.32 \times 1.27)/(-196.29 + 248.32) = 6.07$. Notice that, because a call option is equivalent to a levered position in the stock, it is always riskier

**FIGURE 21.5**

The curved line shows how the value of the Google call option changes as the price of Google stock changes.

eXcel

Visit us at www.mhhe.com/bma




than the stock itself. In Google's case the option is nearly five times as risky as the stock. As time passes and the price of Google stock changes, the risk of the option will also change.

**Some More Practice** Suppose you repeat the calculations for the Google call for a wide range of stock prices. The result is shown in Figure 21.5. You can see that the option values lie along an upward-sloping curve that starts its travels in the bottom left-hand corner of the diagram. As the stock price increases, the option value rises and gradually becomes parallel to the lower bound for the option value. This is exactly the shape we deduced in Chapter 20 (see Figure 20.10).

The height of this curve of course depends on risk and time to maturity. For example, if the risk of Google stock had suddenly decreased, the curve shown in Figure 21.5 would drop at every possible stock price.

### The Black–Scholes Formula and the Binomial Method

Look back at Table 21.1 where we used the binomial method to calculate the value of the Google call. Notice that, as the number of intervals is increased, the values that you obtain from the binomial method begin to snuggle up to the Black–Scholes value of $52.04.

The Black–Scholes formula recognizes a continuum of possible outcomes. This is usually more realistic than the limited number of outcomes assumed in the binomial method. The formula is also more accurate and quicker to use than the binomial method. So why use the binomial method at all? The answer is that there are many circumstances in which you cannot use the Black–Scholes formula but the binomial method will still give you a good measure of the option's value. We will look at several such cases in Section 21-5.

## 21-4 Black–Scholes in Action

To illustrate the principles of option valuation, we focused on the example of Google's options. But financial managers turn to the Black–Scholes model to estimate the value of a variety of different options. Here are four examples.

## Executive Stock Options

The 2007 winner in *Forbes*'s annual list of the most highly paid executives was Larry Ellison, the CEO of Oracle Corporation. His salary was just $1 million, but he also pocketed another $182 million from the exercise of stock options.

The example highlights that executive stock options are often an important part of compensation. For many years companies were able to avoid reporting the cost of these options in their annual statements. However, they must now treat options as an expense just like salaries and wages, so they need to estimate the value of all new options that they have granted. For example, Oracle's financial statements show that in fiscal 2008 the company issued a total of 69 million options with an average life of five years and an exercise price of $20.49. Oracle calculated that the average value of these options was $7.53. How did it come up with this figure? It just used the Black–Scholes model assuming a standard deviation of 29% and an interest rate of 4.6%.[14]

Some companies have disguised how much their management is paid by backdating the grant of an option. Suppose, for example, that a firm's stock price has risen from $20 to $40. At that point the firm awards its CEO options exercisable at $20. That is generous but not illegal. However, if the firm pretends that the options were *actually* awarded when the stock price was $20 and values them on that basis, it will substantially understate the CEO's compensation.[15] The nearby box discusses the backdating scandal.

Speaking of executive stock options, we can now use the Black–Scholes formula to value the option packages you were offered in Section 20-3 (see Table 20.3). Table 21.2 calculates the value of the options from the safe-and-stodgy Establishment Industries at $5.26 each. The options from risky-and-glamorous Digital Organics are worth $7.40 each. Congratulations.

## Warrants

When Owens Corning emerged from bankruptcy in 2006, the debtholders became the sole owners of the company. But the old stockholders were not left entirely empty handed. They were given warrants to buy the new common stock at any point in the next seven

| | Establishment Industries | Digital Organics |
|---|---|---|
| Stock price (P) | $22 | $22 |
| Exercise price (EX) | $25 | $25 |
| Interest rate ($r_f$) | .04 | .04 |
| Maturity in years (t) | 5 | 5 |
| Standard deviation ($\sigma$) | .24 | .36 |
| $d_1 = \log[P/PV(EX)]/\sigma\sqrt{t} + \sigma\sqrt{t}/2$ | 0.3955 | 0.4873 |
| $d_2 = d_1 - \sigma\sqrt{t}$ | −0.1411 | −0.3177 |
| Call value = $[N(d_1) \times P] - [N(d_2) \times PV(EX)]$ | $5.26 | $7.40 |

**TABLE 21.2** Using the Black–Scholes formula to value the executive stock options for Establishment Industries and Digital Organics (see Table 20.3).

eXcel Visit us at www.mhhe.com/bma

[14] Many of the recipients of these options may not have agreed with Oracle's valuation. First, the options were less valuable to their owners if they created substantial undiversifiable risk. Second, if the holders planned to quit the company in the next few years, they were liable to forfeit the options. For a discussion of these issues see J. I. Bulow and J. B. Shoven, "Accounting for Stock Options," *Journal of Economic Perspectives* 19 (Fall 2005), pp. 115–134.

[15] Until 2005 companies were obliged to record as an expense any difference between the stock price when the options were granted and the exercise price. Thus, as long as the options were granted at-the-money (exercise price equals stock price), the company was not obliged to show any expense.

## FINANCE IN PRACTICE

# The Perfect Payday*

On an October day in 1999 the shares of the giant insurer United Health Group sank to their lowest level of the year. That may have been bad news for investors but it was good news for William McGuire, the chief executive, for the company granted him options to buy the stock in the future at that low price. If the options had been dated a month later when the stock price was 40% higher, those options would have been far less valuable. Lucky coincidence? Possibly, but the following year also Mr. McGuire was granted options on the day that the stock price hit the year's low. And in 2001 the grant came near the bottom of a sharp dip in the stock price.

Over the following years evidence began to accumulate that in other companies too executives were being granted options at unusually favorable prices. It seemed that these firms were using hindsight to choose the date on which the options were granted. Such backdating is not necessarily illegal, but most options are granted under a shareholder-approved plan that typically requires the exercise price to be equal to the fair market value of the company's stock at the time of the grant. Also backdating may result in an underestimate of the amount of compensation paid and therefore to a misstatement of earnings and an underpayment of taxes.

Investigations by the SEC and prosecutions by disgruntled shareholders led to the resignation of a number of directors and officers of major corporations that were found to have backdated options. William McGuire was among those who fell on their sword. He subsequently agreed to pay $39 million and forfeit another 3.7 million compensatory stock options to settle a class-action suit headed by the California Public Employee Retirement System (Calpers).

*"The Perfect Payday" is the title of an article in *The Wall Street Journal* that drew attention to the practice of backdating. See C. Forelle and J. Bandler, "The Perfect Payday; Some CEOs Reap Millions by Landing Stock Options When They Are Most Valuable; Luck—or Something Else?" *The Wall Street Journal,* March 18, 2006, p. A1. Earlier evidence of backdating appeared in D. Yermack, "Good Timing: CEO Stock Option Awards and Company News Announcements," *Journal of Finance* 52 (1997), pp. 449–476, and in E. Lie, "On the Timing of CEO Stock Option Awards," *Management Science* 51 (2005), pp. 802–812.

years for $45.25 a share. Because the stock in the restructured firm was worth about $30 a share, the stock needed to appreciate by 50% before the warrants would be worth exercising. However, this option to buy Owens Corning stock was clearly valuable and shortly after the warrants started trading they were selling for $6 each. You can be sure that before shareholders were handed this bone, all the parties calculated the value of the warrants under different assumptions about the stock's volatility. The Black–Scholes model is tailor-made for this purpose.

### Portfolio Insurance

Your company's pension fund owns an $800 million diversified portfolio of common stocks that moves closely in line with the market index. The pension fund is currently fully funded, but you are concerned that if it falls by more than 20% it will start to be underfunded. Suppose that your bank offers to insure you for one year against this possibility. What would you be prepared to pay for this insurance? Think back to Section 20-2 (Figure 20.6), where we showed that you can shield against a fall in asset prices by buying a protective put option. In the present case the bank would be selling you a one-year put option on U.S. stock prices with an exercise price 20% below their current level. You can get the value of that option in two steps. First use the Black–Scholes formula to value a call with the same exercise price and maturity. Then back out the put value from put–call parity. (You will have to adjust for dividends, but we'll leave that to the next section.)

## FINANCE IN PRACTICE

### The Fear Index*

The Market Volatility Index or VIX measures the volatility that is implied by near-term Standard & Poor's 500 Index options and is therefore an estimate of expected *future* market volatility over the next 30 calendar days. Implied market volatilities have been calculated by the Chicago Board Options Exchange (CBOE) since January 1986, though in its current form the VIX dates back only to 2003.

Investors regularly trade volatility. They do so by buying or selling VIX futures and options contracts. Since these were introduced by the Chicago Board Options Exchange (CBOE), combined trading activity in the two contracts has grown to more than 100,000 contracts per day, making them two of the most successful innovations ever introduced by the exchange.

Because VIX measures investor uncertainty, it has been dubbed the "fear index." The market for index options tends to be dominated by equity investors who buy index puts when they are concerned about a potential drop in the stock market. Any subsequent decline in the value of their portfolio is then offset by the increase in the value of the put option. The more that investors demand such insurance, the higher the price of index put options. Thus VIX is an indicator that reflects the price of portfolio insurance.

Between January 1986 and April 2009 the VIX has averaged 20.5%, almost identical to the long-term level of market volatility that we cited in Chapter 7. The high point for the index was in October 1987 when the VIX closed the month at 61%,** but there have been several other short-lived spikes, for example, at the time of Iraq's invasion of Kuwait and the subsequent response by U.N. forces.

Although the VIX is the most widely quoted measure of volatility, volatility measures are also available for several other U.S. and overseas stock market indexes (such as the FTSE 100 Index in the U.K. and the CAC 40 in France), as well as for gold, oil, and the euro.

*For a review of the VIX index see R. E. Whaley, "Understanding the VIX," *Journal of Portfolio Management* 35 (Spring 2009), pp. 98–105.

**On October 19, 1987 (Black Monday), the VIX closed at 150. Fortunately, the market volatility returned fairly rapidly to less exciting levels.

## Calculating Implied Volatilities

So far we have used our option pricing model to calculate the value of an option given the standard deviation of the asset's returns. Sometimes it is useful to turn the problem around and ask what the option price is telling us about the asset's volatility. For example, the Chicago Board Options Exchange trades options on several market indexes. As we write this, the Standard and Poor's 500 index is 890, while a nine-month at-the-money call on the index is priced at 120. If the Black–Scholes formula is correct, then an option value of 120 makes sense only if investors believe that the standard deviation of index returns is about 40% a year.[16]

The Chicago Board Options Exchange regularly publishes the implied volatility on the Standard and Poor's index, which it terms the VIX (see the nearby box). There is an active market in the VIX. For example, suppose you feel that the implied volatility is implausibly low. Then you can "buy" the VIX at the current low price and hope to "sell" it at a profit when implied volatility has increased.

You may be interested to compare the current implied volatility that we calculated earlier with Figure 21.6, which shows past measures of implied volatility for the Standard and Poor's index and for the Nasdaq index (VXN). Notice the sharp increase in investor uncertainty at the height of the credit crunch in 2008. This uncertainty showed up in the price that investors were prepared to pay for options.

[16] In calculating the implied volatility we need to allow for the dividends paid on the shares. We explain how to take these into account in the next section.

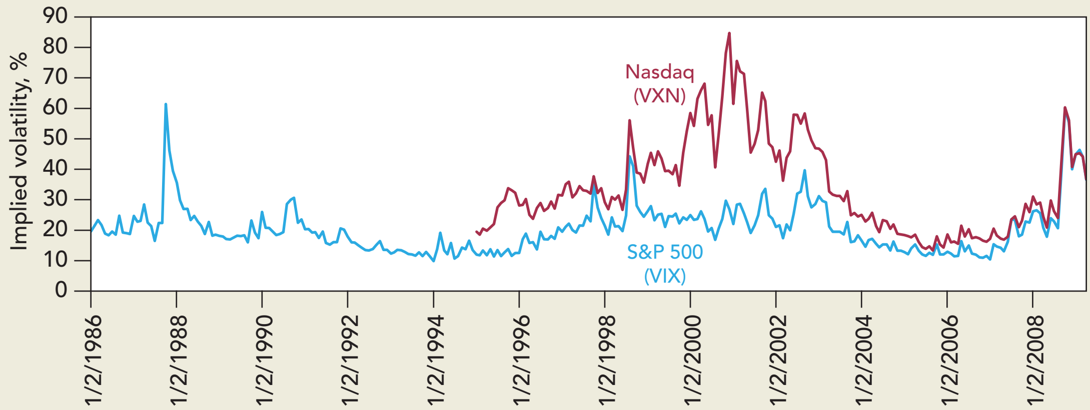



**FIGURE 21.6**

Standard deviations of market returns implied by prices of options on stock indexes.

*Source:* Data from the Chicago Board Options Exchange www.cboe.com.

## 21-5 Option Values at a Glance

So far our discussion of option values has assumed that investors hold the option until maturity. That is certainly the case with European options that *cannot* be exercised before maturity but may not be the case with American options that can be exercised at any time. Also, when we valued the Google call, we could ignore dividends, because Google did not pay any. Can the same valuation methods be extended to American options and to stocks that pay dividends?

Another question concerns dilution. When investors buy and then exercise traded options, there is no effect on the number of shares issued by the company. But sometimes the company itself may give options to key employees or sell them to investors. When these options are exercised, the number of outstanding shares *does* increase, and therefore the stake of existing stockholders is diluted. Can standard option-valuation models cope with the effect of dilution?

In this section we look at how the possibility of early exercise and dividends affect option value. We hold over the problem of dilution to the Appendix to this chapter.

**American Calls—No Dividends** Unlike European options, American options can be exercised anytime. However, we know that in the absence of dividends the value of a call option increases with time to maturity. So, if you exercised an American call option early, you would needlessly reduce its value. Since an American call should not be exercised before maturity, its value is the same as that of a European call, and the Black–Scholes model applies to both options.

**European Puts—No Dividends** If we wish to value a European put, we can use the put–call parity formula from Chapter 20:

$$\text{Value of put} = \text{value of call} - \text{value of stock} + \text{PV(exercise price)}$$

**American Puts—No Dividends** It can sometimes pay to exercise an American put before maturity in order to reinvest the exercise price. For example, suppose that immediately after you buy an American put, the stock price falls to zero. In this case there is no advantage to holding onto the option since it *cannot* become more valuable. It is better to exercise the put and invest the exercise money. Thus an American put is always more valuable than a European put. In our extreme example, the difference is equal to the present value of the interest that you could earn on the exercise price. In all other cases the difference is less.

Because the Black–Scholes formula does not allow for early exercise, it cannot be used to value an American put exactly. But you can use the step-by-step binomial method as long as you check at each point whether the option is worth more dead than alive and then use the higher of the two values.

**European Calls and Puts on Dividend-Paying Stocks** Part of the share value comprises the present value of dividends. The option holder is not entitled to dividends. Therefore, when using the Black–Scholes model to value a European option on a dividend-paying stock, you should reduce the price of the stock by the present value of the dividends to be paid before the option's maturity.

Dividends don't always come with a big label attached, so look out for instances where the asset holder gets a benefit and the option holder does not. For example, when you buy foreign currency, you can invest it to earn interest; but if you own an option to buy foreign currency, you miss out on this income. Therefore, when valuing an option to buy foreign currency, you need to deduct the present value of this foreign interest from the current price of the currency.[17]

**American Calls on Dividend-Paying Stocks** We have seen that when the stock does not pay dividends, an American call option is *always* worth more alive than dead. By holding onto the option, you not only keep your option open but also earn interest on the exercise money. Even when there are dividends, you should never exercise early if the dividend you gain is less than the interest you lose by having to pay the exercise price early. However, if the dividend is sufficiently large, you might want to capture it by exercising the option just before the ex-dividend date.

The only general method for valuing an American call on a dividend-paying stock is to use the step-by-step binomial method. In this case you must check at each stage to see whether the option is more valuable if exercised just before the ex-dividend date than if held for at least one more period.

## 21-6 The Option Menagerie

Our focus in the past two chapters has been on plain-vanilla puts and calls or combinations of them. An understanding of these options and how they are valued will allow you to handle most of the option problems that you are likely to encounter in corporate finance. However, you may occasionally encounter some more unusual options. We are not going to be looking at them in this book, but just for fun and to help you hold your own in

[17] For example, suppose that it currently costs \$2 to buy £1 and that this pound can be invested to earn interest of 5%. The option holder misses out on interest of $.05 \times \$2 = \$.10$. So, before using the Black–Scholes formula to value an option to buy sterling, you must adjust the current price of sterling:

$$\begin{aligned}\text{Adjusted price of sterling} &= \text{current price} - \text{PV(interest)} \\ &= \$2 - .10/1.05 = \$1.905\end{aligned}$$

conversations with your investment banker friends, here is a crib sheet that summarizes a few of these exotic options:

| | |
|---|---|
| Asian (or average) option | The exercise price is equal to the *average* of the asset's price during the life of the option. |
| Barrier option | Option where the payoff depends on whether the asset price reaches a specified level. A knock-in option (up-and-in call or down-and-in put) comes into existence only when the underlying asset reaches the barrier. Knock-out options (down-and-out call or up-and-out put) *cease* to exist if the asset price reaches the barrier. |
| Bermuda option | The option is exercisable on discrete dates before maturity. |
| Caput option | Call option on a put option. |
| Chooser (as-you-like-it) option | The holder must decide before maturity whether the option is a call or a put. |
| Compound option | An option on an option. |
| Digital (binary or cash-or-nothing) option | The option payoff is zero if the asset price is the wrong side of the exercise price and otherwise is a fixed sum. |
| Lookback option | The option holder chooses as the exercise price any of the asset prices that occurred before the final date. |
| Rainbow option | Call (put) option on the best (worst) of a basket of assets. |

## SUMMARY

In this chapter we introduced the basic principles of option valuation by considering a call option on a stock that could take on one of two possible values at the option's maturity. We showed that it is possible to construct a package of the stock and a loan that would provide exactly the same payoff as the option *regardless* of whether the stock price rises or falls. Therefore the value of the option must be the same as the value of this replicating portfolio.

We arrived at the same answer by pretending that investors are risk-neutral, so that the expected return on every asset is equal to the interest rate. We calculated the expected future value of the option in this imaginary risk-neutral world and then discounted this figure at the interest rate to find the option's present value.

The general binomial method adds realism by dividing the option's life into a number of subperiods in each of which the stock price can make one of two possible moves. Chopping the period into these shorter intervals doesn't alter the basic method for valuing a call option. We can still replicate the call by a package of the stock and a loan, but the package changes at each stage.

Finally, we introduced the Black–Scholes formula. This calculates the option's value when the stock price is constantly changing and takes on a continuum of possible future values.

An option can be replicated by a package of the underlying asset and a risk-free loan. Therefore, we can measure the risk of any option by calculating the risk of this portfolio. Naked options are often substantially more risky than the asset itself.

When valuing options in practical situations there are a number of features to look out for. For example, you may need to recognize that the option value is reduced by the fact that the holder is not entitled to any dividends.

## FURTHER READING

*Three readable articles about the Black–Scholes model are:*

F. Black, "How We Came up with the Option Formula," *Journal of Portfolio Management* 15 (1989), pp. 4–8.

F. Black, "The Holes in Black–Scholes," *RISK Magazine* 1 (1988), pp. 27–29.